

Ground State Entanglement Energetics

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Abstract

We consider the ground state of simple quantum systems coupled to an environment. In general the system is entangled with its environment. As a consequence, even at zero temperature, the energy of the system is not sharp: a projective measurement can find the system in an excited state. We show that energy fluctuation measurements at zero temperature provide entanglement information. For two-state systems which exhibit a persistent current in the ground state, energy fluctuations and persistent current fluctuations are closely related. The harmonic oscillator serves to illustrate energy fluctuations in a system with an infinite number of states. In addition to the energy distribution we discuss the energy-energy time-correlation function in the zero-temperature limit.

Key words: entanglement energetics, energy fluctuations, persistent currents, decoherence

1. Introduction

A quantum system cooled to zero temperature nevertheless knows about its environment since generically the system state and the bath state are entangled. The ground state does not factorize into a product of a system wave function and a bath wave function. Entanglement of two subsystems [1] is often discussed in terms of the strange non-local properties it implies for systems that can be spatially separated. Here we consider two-level systems, often now called qubits, or harmonic oscillators which are coupled to a bath. In such a "thermodynamic" setting we can not easily separate the two systems and apply a Bell test [2] to verify the entanglement. Nevertheless, systems entangled with reservoir states exhibit a number of properties which distin-

guishes them from systems for which the ground state factorizes.

A particularly instructive quantity to consider is the energy of the system. First, from a purely theoretical point of view, we always have the energy of the system as an observable. We write the energy of the total system as

$$\hat{H} = \hat{H}_s + \hat{H}_c + \hat{H}_b \quad (1)$$

where \hat{H}_s is the system, \hat{H}_c the coupling and \hat{H}_b the bath energy operator. A second, more important reason for considering the energy is that at zero temperature fluctuations in energy are a direct indicator for system bath entanglement [3]. In contrast, if the system-bath state is not entangled, the system is simply in its lowest energy state. This case is often viewed as self-evident instead of the generic case addressed here.

Consider a two level system with energies E_+ and E_- and probabilities p_{\pm} to find the system in the excited level and in its ground state. The expectation value of the energy is

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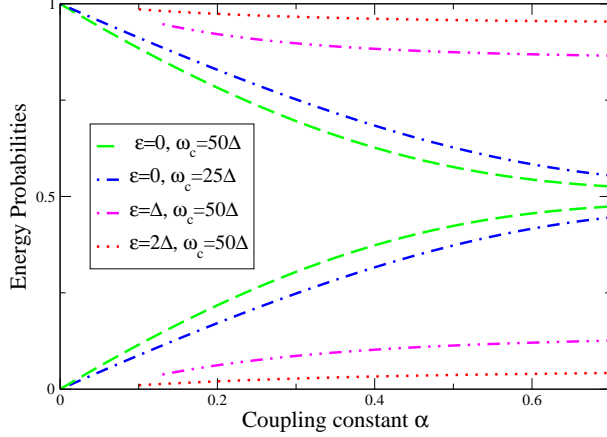


Fig. 1. Energy probabilities p_+ and p_- for the spin-boson problem as a function of coupling strength α and different ϵ . For weak coupling to the environment increasing coupling increases the the probability to find the qubit in the excited state. with increasing coupling to the environment it is more likely to measure the qubit in the excited state. After Ref.[3]

$$\langle \hat{H}_s \rangle = E_- p_- + E_+ p_+ \quad (2)$$

and the energy fluctuation away from its average is

$$\langle (\hat{H}_s - \langle \hat{H}_s \rangle)^2 \rangle = (E_+ - E_-)^2 p_- p_+. \quad (3)$$

For a moment consider the system to be in thermal equilibrium coupled to a bath at temperature T . Later we will only consider the zero-temperature limit. Assume for the moment that there is no entanglement. Then statistical mechanics tells us that $p_+ \propto \exp(-E_+/kT)$ and $p_- \propto \exp(-E_-/kT)$ or with proper normalization ($p_+ + p_- = 1$)

$$p_{\pm} = \frac{1}{1 + \exp(\pm(E_+ - E_-)/kT)}. \quad (4)$$

As the temperature tends to zero, p_- tends to one (p_+ tends to zero) and consequently the energy fluctuations Eq. (3) tend to zero. The energy of the system is a sharp variable in the ground state.

This simple picture is dramatically modified if the system and the bath are entangled. Now statistical mechanics must be applied to the entire system (system plus bath plus interaction energy). The system alone is now determined by a (reduced) density matrix with elements $\rho_{\pm, \pm}$ on the diagonal and non-diagonal elements $\rho_{\pm, \mp}$. In the energy eigen basis of the two state system, $p_{\pm} = \rho_{\pm, \pm}$. While the ground state of the entire system is in general a pure state, the density matrix

of the system alone is in general that of a mixed state. For the simple model of a two state system (a spin) coupled to a harmonic oscillator bath (the spin-boson problem [4,5]) an ohmic bath leads for weak coupling to a probability [3]

$$p_+ = \alpha \log(\omega_c/\Omega). \quad (5)$$

Here α is the system bath coupling constant, $\hbar\Omega = E_+ - E_-$ is the energy separation of the two levels and ω_c is a cut-off of the oscillator spectrum of the bath (a Debye frequency). As a consequence, even at zero temperature the energy of the system is not sharp but fluctuates according to Eq. (3)

$$\langle (\hat{H}_s - \langle \hat{H}_s \rangle)^2 \rangle = \alpha(\hbar\Omega)^2 \log(\omega_c/\Omega). \quad (6)$$

We emphasize that this result is not special for a two state system. For instance for a harmonic oscillator with frequency ω_0 coupled to an ohmic bath of harmonic oscillators, we find that the energy fluctuations[6,7] are given by

$$\langle (\hat{H}_s - \langle \hat{H}_s \rangle)^2 \rangle = \alpha(\hbar\omega_0)^2 \log(\omega_c/\omega_0). \quad (7)$$

Thus $\hbar\omega_0$ plays a role similar to the energy separation $\hbar\Omega$ of the two level system.

The energy distribution is certainly far from being Gaussian. Thus the mean square deviations given by Eqs. (6) and (7) might not be a good indicator of the way the energy is distributed over the different states of the system. For instance for the two state system discussed above the probability distribution $P(E)$ to find the system with energy E in the interval dE obviously consists of two *delta-function* peaks at E_+ and E_- . Thus we can write this distribution in the form,

$$P(E) = p_+ \delta(E - E_+) + p_- \delta(E - E_-) \quad (8)$$

where p_{\pm} are as above the probabilities to find the system in the excited state and in the lowest energy state of the system. Now since $p_- + p_+ = 1$ and since $\langle E \rangle \equiv \langle H_s \rangle = E_- p_- + E_+ p_+$ we can also express the probabilities p_{\pm} in the form $p_+ = \langle E \rangle / \hbar\Omega - E_- / \hbar\Omega$ and $p_- = E_+ / \hbar\Omega - \langle E \rangle / \hbar\Omega$. Here we have used $\hbar\Omega = E_+ - E_-$. Without loss of generality we can set $E_{\pm} = \pm \hbar\Omega/2$ and thus find for the distribution [3]

$$P(E) = \frac{1}{2} \left(1 + \frac{\langle E \rangle}{2\hbar\Omega} \right) \delta\left(E - \frac{\hbar\Omega}{2}\right) + \frac{1}{2} \left(1 - \frac{\langle E \rangle}{2\hbar\Omega} \right) \delta\left(E + \frac{\hbar\Omega}{2}\right) \quad (9)$$

Note that if the system and bath are decoupled we have $\langle E \rangle = -\hbar\Omega/2$ and the distribution function consists of only one peak at $E = -\hbar\Omega/2$.

We emphasize that $\langle E \rangle \equiv \langle H_s \rangle$ is the expectation value of the system's energy in the overall ground state of the system plus bath plus interaction energy. Below we will also discuss $P(E)$ for the harmonic oscillator. For a harmonic oscillator $P(E)$ consists of an infinite number of *delta*-functions with a rapidly decreasing weight of the higher lying states.

To be specific, consider the two-state system Hamiltonian to be

$$H_s = (\epsilon/2) \sigma_z + (\Delta/2) \sigma_x \quad (10)$$

where ϵ measures the distance from resonance and Δ is the energy separation at resonance. σ_z and σ_x are Pauli spin matrices. The level separation is thus determined by the frequency $\Omega = \sqrt{\epsilon^2 + \Delta^2}/\hbar$. If this system is coupled via σ_z to a harmonic oscillator bath (spin-boson problem) H_s will not commute with the total Hamiltonian. For this system the probabilities to find the system in the excited state and in the low energy state are shown in Fig. (1). At resonance $\epsilon = 0$, the probabilities p_{\pm} tend with increasing coupling constant towards 1/2. For these parameters we can use the Bethe solution of the anisotropic Kondo model [8,9]. In the strong coupling limit an energy measurement will find the system with equal probability in the ground state and in the excited state of the system. If the system is not symmetric $\epsilon \neq 0$ the probability to find the system in an excited state reaches a maximum as a function of α and tends to zero for very strong coupling. For large ϵ the probabilities can be found from perturbation theory [8,9]. Combining both the Bethe Ansatz solutions and the perturbation theory [9] it is possible to give the probabilities over the entire range of parameters. Computational work based on renormalization is reported in Ref. [10].

Experiments are always carried out at finite temperature, and it is important to demonstrate that there exists a cross-over temperature to the quantum behavior discussed here. In the low temperature limit, the thermal occupation probability is $p_+ = e^{-\hbar\Omega/kT}$ (see Eq. (4)). In the weak coupling limit for the symmetric spin boson problem, the probability to measure the state as "spin up" scales as $p_+ = \alpha \log(\omega_c/\Omega)$. Setting these factors equal and solving for T^* yields [3]

$$kT^* = -\frac{\hbar\Omega}{\log(\alpha \log \frac{\omega_c}{\Omega})}. \quad (11)$$

Since T^* scales as the inverse logarithm of the coupling constant, it is experimentally possible to reach a regime where thermal excitation are negligible. Experimentally and theoretically [11] one might be tempted to define temperature with the help of the qubit by fitting $p_+ = e^{-\hbar\Omega/kT_{eff}}$ with an effective temperature to the experimental data. Experimentally one would then find that it is impossible to cool the qubit below the temperature T^* . Still a careful examination would show that the state is not in fact "thermal" since it depends on the coupling constant to the bath.

The temperature T^* can be viewed as a measure of the energy difference between the energy of the lowest energy separable state and the true entangled ground state. Recent works emphasize the role of this energy difference as an entanglement witness [12,13]. We now relate energy fluctuations directly to entanglement in the ground state.

2. Energy fluctuations as an entanglement witness

Energy fluctuations are determined by probabilities alone, that is by the diagonal matrix elements of the density matrix only. Thus it is not obvious that we can make a statement about entanglement. In general such a statement also depends on the non-diagonal elements of the density matrix. However, we are given the additional information that we are in the ground state. If we measure the subsystem's energy and find an excited energy, then we know that the state is entangled. For weak coupling we can make quantitative statement. The probability to find the system in higher lying states is exponentially suppressed. To first order in the coupling constant, we can consider a two-state system where the density matrix has the form $p_- \equiv \rho_{--} = 1 - \alpha p$, $p_+ \equiv \rho_{++} = \alpha p$, $\rho_{+-} = \rho_{-+}^* = \alpha c$.

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha \begin{pmatrix} -p & c \\ c^* & p \end{pmatrix} + O(\alpha^2). \quad (12)$$

For vanishing coupling constant $\alpha = 0$, this just gives the density matrix for the separable state. The linear dependence of ρ on α holds to first order for the model

systems considered here and is the entanglement contribution. If one measures the diagonal elements of ρ , one obtains ρ_{--} and ρ_{++} as the probability to measure the system in the ground or excited state. If we now diagonalize ρ , the eigenvalues are $\lambda_{\pm} = \{1 - p\alpha, p\alpha\} + O(\alpha^2)$. To first order in α , the eigenvalues are the diagonal matrix elements, so we may (to a good approximation) write entanglement measures like the purity in terms of these probabilities. As an example the purity $Tr(\rho^2)$ is given by $Tr(\rho^2) = \lambda_+^2 + \lambda_-^2 \sim 1 - 2p\alpha = 1 - 2p_+$.

We next consider two examples of qubits which exhibit the behavior discussed above.

3. Persistent current qubits

Thus far we have focused on the energy of the system as the quantity of interest. In this section we show that other observables which reflect properties of the system behave in fact very similarly. In particular we discuss two states systems (qubits) which have the property that their tunnel matrix element Δ in Eq. (10) is dependent on an Aharonov-Bohm flux Φ . The free energy of such a system then depends on the flux and in general supports a persistent current in its ground state $I(\Phi) = -dF/d\Phi$. The prediction of persistent currents in small normal and disordered loops [14] has played an important role in the development of mesoscopic physics and continues to be a subject of current interest [16].

3.1. The mesoscopic persistent current qubit

A small metallic loop shown in Fig. (2) can be made into a two state system with the help of a quantum dot [17]. For sufficiently small charging energy only the states with N and $N + 1$ electrons on the dot will be relevant. As long as these two states are energetically very different, charge on the dot is fixed and transport through the dot is blocked. Only near the point of degeneracy can electrons tunnel in and out of the dot. In the charge basis, the charge on the dot is proportional to σ_z with an energy ϵ that determines how far away the system is from the point of degeneracy. At the point of degeneracy, charge tunneling is permitted since the dot is weakly coupled with tunnel energies t_L

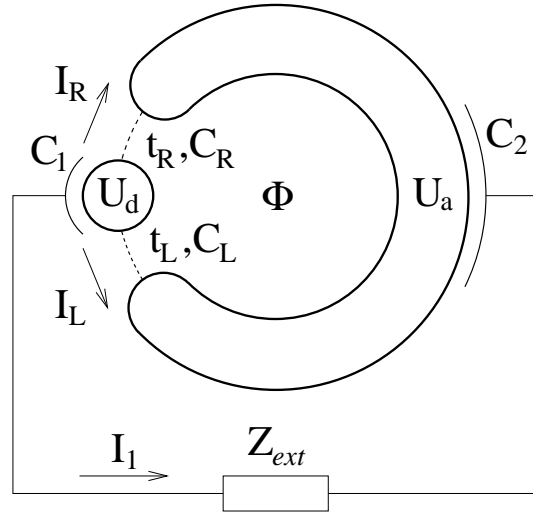


Fig. 2. Ring with an in-line quantum dot penetrated by an Aharonov-Bohm flux. The ring is coupled capacitively to an external circuit. After Ref.[20]

and t_R to its contacts. In such a ring the persistent current exhibits sharp peaks at special values of the gate voltage [17] much like the Coulomb peaks of conductance. The Aharonov-Bohm flux can be incorporated into an effective tunnel matrix element which connects the charge states N and $N + 1$,

$$\Delta^2/4 = t_L^2 + t_R^2 \pm 2t_L t_R \cos(2\pi\Phi/\Phi_0). \quad (13)$$

Here the sign depends on the number of electrons in the ring. Thus near a point of degeneracy the Hamiltonian of this ring is of the form given by Eq. (10). We refer the reader to Refs. [17] for a detailed derivation.

The system is coupled to an external circuit via capacitances. In particular, if the external circuit is an ohmic resistor, we can replace it with a transmission line with equal input impedance [9]. The transmission line represents a harmonic oscillator bath [18] and the entire system is a particular realization of the spin-boson problem with the interesting feature that the tunnel matrix-element is flux dependent.

The model permits us to address the interesting question of how persistent currents are affected by environments. We follow here the discussion of Cedraschi, Ponomarenko and one of the authors [9]. An extended discussion of the initial work is provided in Ref. [19]. The persistent current is obtained using the results of Bethe Ansatz and perturbation results for the anisotropic Kondo model [8]. For weak system-

bath coupling, Ref. [20] provides a discussion based on a quantum Langevin approach. Closely related works investigate the effect of a fluctuating Aharonov-Bohm flux [21], the effect of charge fluctuations in nearby pure and disordered conductors [22,23] and the effect of hot bosonic baths [24].

Let us now first show that the persistent current, like the energy of the system, is not sharp but fluctuates. We are interested in current fluctuations so we need an expression for the current operator. Since the persistent current is due to electrons of the ring alone we can consider the isolated system $\alpha = 0$. Eq.(10) is the Hamiltonian in the charge basis. In the eigen basis the Hamiltonian is simply $\hat{H}_s = (\hbar\Omega/2)\sigma_z$. In the eigen basis the persistent current carried by a state is determined by the derivative of the energy of this state with respect to flux. The persistent current of the eigenstates with energies $\pm\hbar\Omega/2$ is $\mp I_0$ with $I_0 = (1/2)d\hbar\Omega/d\Phi$. One of the states corresponds to a clockwise persistent current and one state corresponds to a counter-clockwise current. Thus in the eigen basis the current operator is simply

$$\hat{I} = I_0\sigma_z = -(2I_0/\hbar\Omega)\hat{H}_s. \quad (14)$$

Here we use that in the eigen basis, both the current operator and the energy are proportional to σ_z . Thus the persistent current is directly related to the energy of the system [25]. While the first expression in Eq. (14) is valid only in the energy eigen state basis, the second expression is in fact general.

In the presence of a bath, the persistent current, like the energy of the two state system, is in fact not sharp but fluctuates. Since the average persistent current is $\langle\hat{I}\rangle = I_+p_+ + I_-p_-$ with $p_+ + p_- = 1$ we can also express p_{\pm} in terms of the persistent current. With $I_{\pm} = \pm I_0$ we find $p_{\pm} = (1/2)(1 \mp \langle\hat{I}\rangle/I_0)$ and the distribution of currents is thus

$$P(I) = \frac{1}{2}(1 + \frac{\langle\hat{I}\rangle}{I_0})\delta(I - I_0) + \frac{1}{2}(1 - \frac{\langle\hat{I}\rangle}{I_0})\delta(I + I_0) \quad (15)$$

Note that $p_{\pm} = (1/2)(1 \mp \langle\hat{I}\rangle/I_0) \leq 1$ implies that coupling the ring to the bath can only suppress [9] the persistent current $\langle\hat{I}\rangle \leq I_0$.

Next let us investigate the flux dependence in more detail. In the absence of coupling to a bath $\alpha = 0$, and at the point of degeneracy $\epsilon = 0$ the persistent current is, $I_0(\Phi) = d\Delta(\Phi)/d\Phi$. Consider now additionally the special symmetric case, when the tunneling rates are

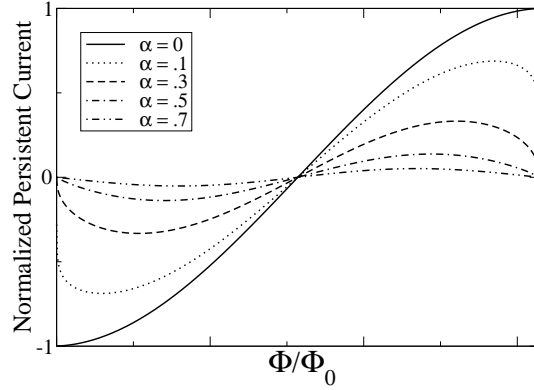


Fig. 3. Persistent current as a function of the Aharonov-Bohm flux at resonance ($\epsilon = 0$) for different coupling strengths α . Arbitrary small coupling suppresses the discontinuity of the $\alpha = 0$ -persistent current at $\Phi/\Phi_0 = 0$ and 1.

equal $t \equiv t_L = t_R$. Taking the lower sign in Eq. (13) the persistent current is $I(\Phi) = -(e/h)4\pi t \cos(\pi\Phi/\Phi_0)$ in the interval $0 \leq \Phi \leq \Phi_0$. It is a periodic function with a discontinuous jump at $\Phi = n\Phi_0$, $n = 0, \pm 1, \pm 2, \dots$ It is shown as a solid line in Fig. (3).

It is now very interesting to investigate what happens to the discontinuous jump in the persistent current in the presence of the bath. Since a Fourier representation of the persistent current

$$I(\Phi) = \sum_n I_n \sin(2\pi n\Phi/\Phi_0) \quad (16)$$

needs arbitrary high harmonics such a jump is a signature of a perfectly coherent system: the top-most electron in our system which gives rise to this current must circulate the Aharonov-Bohm flux coherently n -times to generate the n -th harmonic. Coupling to a bath generates de-coherence, and we suspect that the bath suppresses such a discontinuity immediately. The Bethe Ansatz solution of the spin-boson model gives a persistent current

$$I(\Phi) \propto \Delta^{(\frac{\alpha}{1-\alpha})} d\Delta/d\Phi \quad (17)$$

where α is the coupling constant. The exact result is given by Cedraschi et al. [9]. The persistent current as a function of flux for different coupling constants is shown in Fig. (3). Thus an arbitrary small coupling to a bath is sufficient to suppress the discontinuity in the persistent current. Using Eq. (17) a more quantitative analysis can be provided. Pilgram [26] finds for the Fourier amplitudes

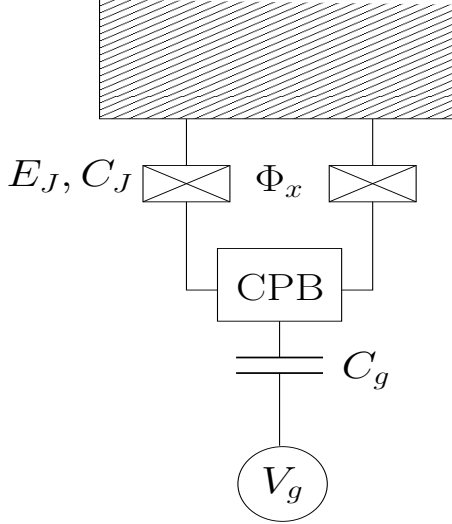


Fig. 4. Cooper pair box: A superconducting metallic dot is coupled via two Josephson junctions (with energy E_J and capacitance C_J) to a superconductor terminal and is capacitively coupled (with capacitance C_g) to a gate permitting the control of charge. An external flux Φ_x through the hole of the structure controls the Josephson energy.

$$\frac{I_n}{I_1} = n \frac{(2\alpha - 1) \dots ((2n - 2)\alpha - (2n - 3))}{(4\alpha - 5) \dots ((2n\alpha - (2n + 1)))} \quad (18)$$

With the Ansatz

$$\frac{I_n}{I_1} = A_n \exp(-b_n(\alpha)\alpha(n - 1)) \quad (19)$$

the values found for $b_n(\alpha)$ are, $b_2 = 6/5$, $b_3 = 88/105$, $b_4 = 626/945 \dots$. Thus the bath suppresses the persistent current almost in an exponential manner, as if the system were subject to dephasing. This suppression is the stronger the higher the harmonics.

The system of Fig. (2) can be viewed as a double quantum dot. Recent experimental work by Hayashi et al. [27] has demonstrated that conditions can be achieved for which the double dot system is well described by a Hamiltonian of the form Eq. (10). The work also demonstrates how strong measurements can be implemented. Refs. [28,29] represent closely related theoretical work.

3.2. The split Cooper pair box qubit

In this section we compare briefly a superconducting structure [30,31,32,33] with a behavior that is analogous to the model discussed above. This example is

important since, in contrast to the normal state qubit discussed above, it does not depend on single particle energies and since a projective measurement testing the state of the system by measuring its persistent current, has in fact been implemented [33]. A review of the current research on superconducting qubits is provided by Devoret, Walraff and Martinis [34]. The structure of interest here is a Cooper pair box [35], a small superconducting island coupled with tunnel junctions to a large superconductor and coupled capacitively to a gate (see Fig. 4). The small superconducting island can be split [32] such that it forms together with the large superconductor a ring. If the capacitances of the junctions C_J and the capacitance to the gate C_g are small, the energy for charging the island with an additional Cooper pair $E_C = (2e)^2/(2C_J + C_g)$ is large and the system can effectively be described with a two-state Hamiltonian. The limit of interest is the charge controlled Cooper pair box [34] for which the Josephson energy E_J is much smaller than the charging energy E_C . The parameters [34] of the two-state Hamiltonian Eq. (10) are $\epsilon = E_J \cos(\pi\Phi_x/\Phi_0)$ where Φ_x is the externally applied flux, $\Phi_0 = h/2e$ the charge $2e$ -flux quantum and $\Delta = E_C(1/2 - N_g)$ where N_g can be controlled by adjusting a gate voltage. Comparing the superconducting qubit to the normal conducting qubit we notice that here it is ϵ which is flux dependent.

We can proceed with the discussion of the persistent current as in the normal case: In the eigen basis of the Hamiltonian the persistent current in the low and higher energy states is $\pm I_0 = \pm d\hbar\Omega/d\Phi$ with $\hbar\Omega/2 = [E_J^2 \cos^2(\pi\Phi_x/\Phi_0) + E_C^2(1/2 - N_g)^2]^{1/2}$. The operator of the persistent current is as for the mesoscopic qubit given by $\hat{I} = -(2I_0/\hbar\Omega)\hat{H}_s$. The persistent current of this superconducting qubit, when it is coupled to a bath (external circuit) fluctuates even in the ground state. In the experiment of Vion et al. [33] Rabi oscillations are reported with an extremely small damping, a Q -factor as high as 25'000. This value can be used to estimate the coupling constant α and leads to a probability to find the system in the excited state of only 10^{-4} . This is too small to be of significance when compared to other effects, due to the measurement circuit or nearby charge traps. The effect we have discussed is most relevant in the strong coupling case between system and bath. Recent experiments in which the Cooper pair box is coupled to a cavity mode [36] offer ways to explore the strong coupling limit of interest here.

4. Energy fluctuations of the harmonic oscillator

We now consider the entanglement energetics of a harmonic oscillator,

$$H_s = p^2/(2m) + (1/2)m\omega^2 q^2. \quad (20)$$

Since there are an infinite number of states, the problem is harder. We assume a linear coupling with a harmonic oscillator bath. As a consequence the density matrix is Gaussian so that environmental information is contained in the second moments $\langle q^2 \rangle$ and $\langle p^2 \rangle$ [5],

$$\langle q|\rho|q' \rangle = \frac{1}{\sqrt{2\pi\langle q^2 \rangle}} \exp \left\{ -\frac{(\frac{q+q'}{2})^2}{2\langle q^2 \rangle} - \frac{\langle p^2 \rangle (q-q')^2}{2\hbar^2} \right\}. \quad (21)$$

Expectation values of higher powers of H_s are non-trivial because q and p do not commute. The purity of the density matrix Eq. (21) is

$$\text{Tr}\rho^2 = \int dq dq' \langle q|\rho|q' \rangle \langle q'|\rho|q \rangle = \frac{\hbar/2}{\sqrt{\langle q^2 \rangle \langle p^2 \rangle}}. \quad (22)$$

The uncertainty relation, $\sqrt{\langle q^2 \rangle \langle p^2 \rangle} \geq \hbar/2$, guarantees that $\text{Tr}\rho^2 \leq 1$ with the inequality becoming sharp if the oscillator is isolated from the environment. As the

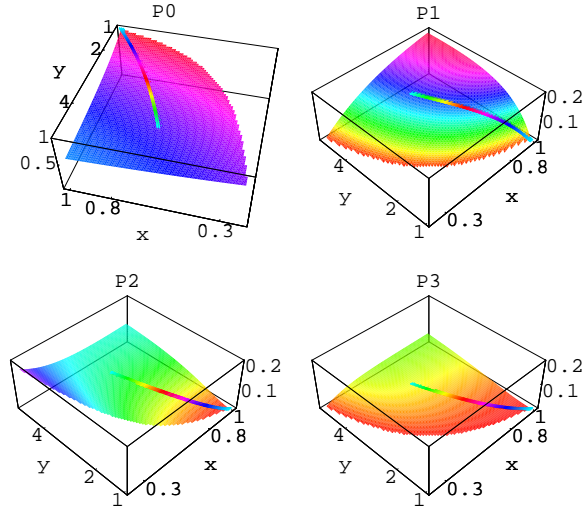


Fig. 5. The probability to measure a harmonic oscillator in the ground and first three excited states as a function of x and y (see text). The line traces out the behavior of the ohmic bath as a function of the coupling in the under-damped range. After [3]

environment causes greater deviation from the Planck scale limit, the state loses purity.

4.1. Energy cumulants

We calculate the generating function $Z(\chi) = \langle \exp(-\chi H_s) \rangle$ from which we can determine the n^{th} energy cumulant

$$\langle \langle H_s^n \rangle \rangle = (-)^n \frac{d^n}{d\chi^n} \ln Z(\chi) \Big|_{\chi=0}. \quad (23)$$

by taking derivatives. Ref. [3] finds

$$Z = \left\{ 2E \frac{\sinh \varepsilon \chi}{\varepsilon} + 2A (\cosh \varepsilon \chi - 1) + \frac{1 + \cosh \varepsilon \chi}{2} \right\}^{-\frac{1}{2}} \quad (24)$$

where $\varepsilon = \hbar\omega$, $2E = m\omega^2 \langle q^2 \rangle + \langle p^2 \rangle/m$ and $A = \langle q^2 \rangle \langle p^2 \rangle / \hbar^2$. E is the average energy of the oscillator, while $A \geq 1$ is a measure of satisfaction of the uncertainty principle.

Using Eq. (23), the first few harmonic oscillator energy cumulants are straightforwardly found via Eq. (23),

$$\langle \langle H_s^2 \rangle \rangle = (1/2)[-(\varepsilon^2/2) + 4E^2 - 2\varepsilon^2 A], \quad (25)$$

$$\langle \langle H_s^3 \rangle \rangle = -(E/2)[-16E^2 + \varepsilon^2(1 + 12A)], \quad (26)$$

$$\langle \langle H_s^4 \rangle \rangle = 48E^4 - 4\varepsilon^2 E^2(1 + 12A) + \varepsilon^4[(1/8) + 2A + 6A^2]. \quad (27)$$

After inserting the mean square values for an ohmic bath (see the discussion above Eqs. (29,30)), Eq. (25) is identical to Eq. (7).

4.2. Density matrix

Alternatively, we now consider the diagonal matrix elements ρ_{nn} . An analytical expression for the density matrix in the energy basis may be found by using the wavefunctions of the harmonic oscillator, $\psi_n(q) \propto e^{-\gamma^2 q^2/2} H_n(\gamma q)$ where $\gamma = \sqrt{m\omega/\hbar}$ and $H_n(x)$ is the n^{th} Hermite polynomial. In the energy basis, the density matrix is given by $\rho_{nm} = \int dq dq' \psi_n^*(q) \langle q|\rho|q' \rangle \psi_m(q')$. We first define the dimensionless variables $x = 2\gamma^2 \langle q^2 \rangle$, $y = 2\langle p^2 \rangle/(\gamma^2 \hbar^2)$, and $D = 1 + x + y + xy$. x and y are related to the major and minor axes of an uncertainty ellipse. The isolated harmonic oscillator (in its ground state) obeys two important properties: minimum uncertainty (in position

and momentum) and equipartition of energy between average kinetic and potential energies. The influence of the environment causes deviations from these ideal behaviors which may be accounted for by introducing two new parameters, $a = (y - x)/D$, $b = (xy - 1)/D$ with $-1 \leq a \leq 1$ and $0 \leq b \leq 1$. The deviation from equipartition of energy is measured by a , while the deviation from the ideal uncertainty relation is measured by b . We find

$$\rho_{nn} = \sqrt{\frac{4}{D}} (b^2 - a^2)^{n/2} P_n \left[\frac{b}{\sqrt{b^2 - a^2}} \right], \quad (28)$$

where $P_n[z]$ are the Legendre polynomials. The probability for the lone oscillator to be measured in an excited state clearly decays rapidly with level number. These probabilities also reveal environmental information. For example, $P_1 = 2b$ and is thus only sensitive to the area of the state, while $P_2 = a^2 + 2b^2$ depends on both the uncertainty and energy asymmetry. Additionally, if we expand the first density matrix eigenvalue [5] with respect to small deviations of x and y , we recover ρ_{11} in agreement with our general argument.

Thus far we have treated x and y as independent variables. In reality the environment the system is coupled to replaces these variables with two functions of the coupling constant. For example, with the ohmic bath [5,7] (in the under-damped limit), the variables are

$$x(\alpha) = \frac{1}{\sqrt{1 - \alpha^2}} \left(1 - \frac{2}{\pi} \arctan \frac{\alpha}{\sqrt{1 - \alpha^2}} \right), \quad (29)$$

$$y(\alpha) = (1 - 2\alpha^2)x(\alpha) + \frac{4\alpha}{\pi} \log \frac{\omega_c}{\omega}, \quad (30)$$

where α is the coupling to the environment in units of the oscillator frequency and ω_c is a high frequency cutoff. This bath information is shown in Fig. (5) with $\omega_c = 10\omega$. The trajectory of the line over the surface shows how the probabilities evolve as the coupling α is increased from 0 to 1. Other kinds of environments would trace out different contours on the probability surface.

4.3. Ground state energy-energy correlations

The expectation value of observables of the system, like the energy \hat{H}_s , the persistent current \hat{I} or their moments, are time-independent in the ground state of the total system. However, this is not true for two-time correlations, like the energy-energy correlation function,

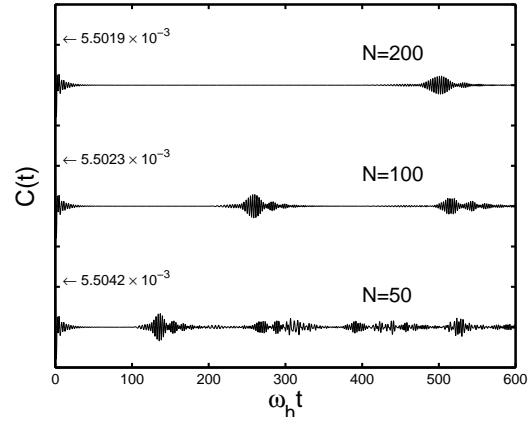


Fig. 6. Energy-energy correlation of a harmonic oscillator coupled to a chain of N particles as a function of time. The correlation is evaluated in the ground state of the entire system. The length of the chain is $N = 50$ (bottom), $N = 100$ (middle), and $N = 200$ (top). After Ref. [7].

$$C(t) \equiv \frac{1}{2} \langle \Delta \hat{H}_s(t) \Delta \hat{H}_s(0) + \Delta \hat{H}_s(0) \Delta \hat{H}_s(t) \rangle. \quad (31)$$

Here $\Delta \hat{H}_s(t) = \hat{H}_s(t) - \langle \hat{H}_s \rangle$ are the energy fluctuations away from the average energy. This correlation function vanishes if the ground state is a product of a system and bath wave function. The correlation is thus also a measurement of the degree of entanglement between system and bath.

For the oscillator, Eq. (20), coupled to a linear chain of N particles with elongation $x_n, n = 1, \dots, N$ coupled with an energy $(1/2)m_h\omega_h^2(x_{n-1} - x_n)^2$ this correlation function was calculated by K. E. Nagaev and one of the authors [7]. An infinitely long chain generates friction proportional to $\eta = (m_h/m)\omega_h$ giving rise to a system bath coupling constant $\alpha = (m_h/m)(\omega_h/\omega)$. The calculation proceeds by first searching the normal modes of the classical problem. The corresponding classical problem is then quantized. The energy of the subsystem and in particular the energy-energy correlation is written in terms of the normal modes of the entire system. The correlation is shown in Fig. (6) as a function of time for three different chain lengths $N = 50, 100, 200$ with $\omega/\omega_h = 1$ and $m_h/m = 0.1$. The numbers on the vertical axis indicate the initial value of the correlation. There is a very rapid initial decay of the correlation followed by a damped oscillatory behavior. Since the chain is of finite length a partial revival is seen after a time it takes a perturbation to travel down the chain and back to the oscillator. We emphasize that the re-

vival is not complete. The spectrum of the chain alone would consist of commensurate frequencies, however, due to the presence of the harmonic oscillator the spectrum of the entire system is not commensurate.

Unlike in the case of the two-state systems we can not easily develop a model for the harmonic oscillator which connects it to persistent currents. However, the conductance in an Aharonov-Bohm geometry in which electrons traverse an oscillatory potential has recently been discussed by Ratchov et al. [37].

5. Discussion

The energy of the subsystem is an observable which illustrates best the distinction between separable and entangled ground states. We have shown that projective measurements of the system Hamiltonian at zero temperature can find the system in higher energy states. This is the case if the many-body quantum mechanical ground state of system and environment are entangled.

Entanglement assures that the system "knows" about its environment and similarly there is system information in the environment. If the environment is represented as a linear chain of particles the system bath interaction can be viewed as a scattering problem. System information is then present in the bath in the form of the phase of the reflected part of the scattering state [38].

We emphasize that the system bath interaction is not just a question of renormalization. For instance in the two state problem discussed here the tunnel matrix element Δ is "renormalized" to Δ_{eff} . However, clearly the physics we have discussed here is not captured by simply replacing in the two-state Hamiltonian the tunnel matrix element by its renormalized value. Both a two-state system or a harmonic oscillator (with renormalized mass and frequency) would exhibit a sharp energy in their ground state.

One natural question is how this work is connected to the presence or absence of dephasing at zero temperature. Historically, dephasing has mainly dealt with the randomizing of a quantum mechanical phase via interaction with some fluctuating variable, such as a reduction of the Aharonov-Bohm interference pattern from voltage fluctuations, which usually freeze out at

low temperature. Typically, the off-diagonal density matrix elements decay in time, while the diagonal elements stay constant.

A more modern view is to call decoherence any mechanism where one starts with a pure system state, and ends with a mixed system state. The ultimate cause of this process is simply entanglement of the system under observation with other degrees of freedom that are unmonitored. Thus, although the entire quantum system may be in a pure state, the fact that local measurements on the subsystem extract only part of the information, results in mixed behavior.

In this sense, there is trivially decoherence at zero temperature, unless either the coupling constant vanishes (so the ground state is separable), or it is possible to make measurements on every coupled quantum degree of freedom, so the purity of the many-body ground state is accessible.

Relaxation into equilibrium is probably the simplest possible preparation method of an entangled state. For this reason ground state entanglement energetics will likely be an important direction of future research.

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